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Monopole and Dyon Bound States in N=2 Supersymmetric Yang-Mills Theories

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We study the existence of monopole bound states saturating the BPS bound in N=2 supersymmetric Yang-Mills theories. We describe how the existence of such bound states relates to the topology of index bundles over the moduli space of BPS solutions. Using an L^2 index theorem, we prove the existence of certain BPS states predicted by Seiberg and Witten based on their study of the vacuum structure of N=2 Yang-Mills theories.

8/95

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1. Introduction

Strong-weak coupling duality, or S-duality, has been conjectured in certain supersymmetric field theories and string theories. The existence of such a symmetry was originally proposed by Montonen and Olive for Yang-Mills theories [1]. Among the testable predictions of S-duality is the existence of certain Bogomol'nyi-Prasad-Sommerfeld (BPS) bound states. For N=4 Yang-Mills, Sen verified the existence of such dyon bound states with magnetic charge two [2]. This provided a strong dynamical test for the existence of S-duality in this theory. For N=2 Yang-Mills theories, Seiberg and Witten have proposed a generalized S-duality involving the dependence of the theory on the Higgs field expectation value [3]. For the case of N=2 Yang-Mills coupled to matter multiplets, they conjecture the existence of certain BPS bound states required by the singularity structure of the vacuum manifold. We propose to verify at least some of their predictions.

Our approach to this problem involves quantizing the low-energy dynamics of N=2 Yang-Mills coupled to matter. The existence of monopole bound states then reduces to a study of the spectrum of the Hamiltonian governing the low-energy dynamics. The existence of a bound state saturating the BPS bound is then equivalent to the existence of a zero mode for a twisted Dirac operator on the monopole moduli space. Since the moduli space is non-compact, we employ an L^2 index theorem to count the number of zero modes, and hence BPS states at magnetic charge two.

We briefly summarize our results. N=2 Yang-Mills with N_f hypermultiplets has no BPS states at magnetic charge two for $N_f < 3$. For $N_f = 3$, we find two bound states: one with allowed electric charges $4n+1$ and the other with allowed charges $4n+3$, where n is an integer.[1] These states are singlets of the $SO(6)$ flavor symmetry. For the candidate S-dual theory with $N_f = 4$, we find bound states corresponding to the $SL(2, \mathbf{Z})$ partners of the fundamental electrons; however, we also find partners for the heavy gauge bosons, implying that they exist as discrete states at threshold in this theory.

In the following section, we derive the Lagrangian governing the low-energy dynamics of monopoles and dyons in supersymmetric Yang-Mills coupled to matter. Section three describes the moduli space of BPS solutions, and the bundles of interest to us. The index computations needed to obtain the BPS spectrum are then presented. Section four provides a comparision with the states predicted by Seiberg and Witten. The final section is a summary and discussion.

[1] The electric charge of an electron is normalized to one as in [3].

2. Supersymmetric Yang-Mills and the Collective Coordinate Expansion

2.1. Pure N=2 Yang-Mills

Collective coordinate expansions for N=2 and N=4 Yang-Mills around BPS monopole configurations have been described in [4]. We require the slightly more general case of N=2 coupled to general matter. Let us denote the $SU(2)$ symmetry rotating the supersymmetry generators by $SU(2)_I$. We shall proceed by first discussing the quantization of zero modes for pure N=2 Yang-Mills and then coupling to matter. In N=1 superspace, the Lagrangian for N=2 Yang-Mills takes the form

$$L_{YM} = \frac{1}{g^2}\int d^4\theta \Phi^\dagger e^{2V}\Phi + \left(\frac{1}{4g^2}\int d^2\theta \mathrm{Tr} W^\alpha W_\alpha + c.c.\right), \tag{2.1}$$

where W^α is a vector multiplet, and Φ a chiral multiplet in the adjoint representation of the gauge group. In terms of component fields, the Lagrangian contains a gauge field A^μ, Higgs field ϕ, and an $SU(2)_I$ doublet of complex Weyl fermions η^j. Note that gauge indices are suppressed for most of this discussion. Let us take the gauge group to be $SU(2)$ which restricts any matter to the fundamental or adjoint representations.

In the Coulomb phase, the gauge group is broken from $SU(2)$ to $U(1)$. The flat directions for the potential correspond to $\left[\phi, \phi^\dagger\right] = 0$. Let ϕ have vacuum expectation value $\left\langle \mathrm{Tr}\,\phi^2 \right\rangle = \frac{1}{2}v^2$ which we choose to be large so that a semi-classical analysis is applicable. In the Coulomb phase, the theory possesses fundamental particles and solitons with masses saturating the BPS bound. The BPS spectrum consists of dyons, W-bosons, and with the inclusion of matter, electrons. Some of these particles may exist at the quantum level. As explained in [3], the mass of such a state is determined by the central extension of the supersymmetry algebra, and is given by

$$M = \sqrt{2}|n_e v + n_m \frac{4\pi i v}{g^2}| \tag{2.2}$$

in the semi-classical limit. Here n_e, n_m are the electric and magnetic charges respectively. This formula remains true when matter is coupled since we shall assume all the electrons have vanishing bare masses.

Since our interest is in checking for the existence of BPS states, we will eventually consider only low-energy fluctuations around a BPS solution. Such fluctuations are tangential to the moduli space of BPS solutions with charge k, and so our computations will reduce to non-relativistic quantum mechanics on the moduli space.

2.2. BPS Field Configurations

We shall follow the conventions of Wess and Bagger [5]. The supersymmetry transformations for the action (2.1) take the form:

$$\begin{aligned} \delta A_\mu &= -i\bar{\eta}_j\bar{\sigma}_\mu\epsilon^j + i\bar{\epsilon}_j\bar{\sigma}_\mu\eta^j \\ \delta\phi &= \sqrt{2}\epsilon^j\eta_j \\ \delta\eta^j &= \sigma^{\mu\nu}F_{\mu\nu}\epsilon^j - i\sqrt{2}\sigma^\mu D_\mu\phi\bar{\epsilon}^j \end{aligned} \tag{2.3}$$

where we have set $\left[\phi, \phi^\dagger\right]$ to zero, and where the ϵ^j_α are anti-commuting parameters. We can further choose ϕ to be real and search for static field configurations which preserve half the supersymmetries. Such configurations satisfy the BPS bound. In the gauge $A_0 = 0$, with the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$, we find the first order BPS equations

$$B_i = \pm\sqrt{2}D_i\phi. \tag{2.4}$$

which also follow immediately from the classical energy for a monopole configuration

$$E = \frac{1}{2g^2}\int (B_i \pm \sqrt{2}D_i\phi)^2 \mp \frac{\sqrt{2}}{g^2}\int \partial_i(B_i\phi). \tag{2.5}$$

The second term is a topological invariant proportional to the magnetic charge and saturating the BPS bound (2.2). For magnetic charge k, a field configuration satisfying (2.4) depends on $4k$ parameters φ^a. These $4k$ moduli are coordinates for an interesting non-compact hyperkähler manifold M_k that will be discussed further in the following section. Associated to each modulus, φ^a, are zero modes (A^i_{oa}, ϕ_{oa}) in the expansion around the monopole configuration (A^i, ϕ). Such zero modes must be orthogonal to the directions generated by infinitesimal gauge transformations with compact support. This condition determines the adjoint-valued parameters ω_a defined by:

$$\begin{aligned} A^i_{0a} &= \partial_a A^i + D^i\omega_a \\ \phi_{oa} &= (\partial_a + \omega_a)\,\phi. \end{aligned} \tag{2.6}$$

The standard technique used to quantize these moduli is to permit the φ^a to vary with time while omitting the zero modes from a perturbative expansion. We will further truncate the mode expansion of (A^i, ϕ) to the moduli dependent classical solution.

We will expand the effective action to second order in the number of time derivatives and later to fourth order in the number of fermion fields. To ensure that the equations of

motion are satisfied to first order, the classical configurations must be corrected by terms of order one; specifically, the constraint $A_0 = 0$ must be modified to satisfy the gauge-field equations of motion where now

$$D_0 = \dot{\varphi}^a \left(\partial_a + \omega_a\right). \tag{2.7}$$

Taking into account such modifications, the bosonic piece of the effective action then describes a sigma model on the moduli space

$$S_{eff} = \int dt \, g_{ab} \dot{\varphi}^a \dot{\varphi}^b, \tag{2.8}$$

where

$$g_{ab} = \frac{1}{g^2} \int d^3x \, (\frac{1}{2} A^i_{oa} A^i_{ob} + \phi_{oa}\phi_{ob}). \tag{2.9}$$

A more detailed discussion is given in [4].

2.3. Fermionic Zero Modes

In the topologically non-trivial monopole background, the fermions possess zero modes. The inclusion of these zero modes in the effective action is readily accomplished by noting that the low-energy theory must be supersymmetric since the BPS configuration preserves a single supersymmetry. The Callias index theorem states that the Dirac operator for fermions in the adjoint representation in a monopole background of charge k has $2k$ zero modes [6]. We must therefore introduce $4k$ real Grassmann collective coordinates γ^a in a mode expansion for the two Majorana fermions in the N=2 gauge multiplet. As in the case of the bosonic moduli, we allow the fermionic moduli to vary with time. By simply counting degrees of freedom, and using the pairing of bosonic and fermionic degrees of freedom required by (2.3), we can easily determine the effective action. The action is most simply expressed in terms of the superfields $\Phi^a = \varphi^a + \theta\gamma^a$ where θ is a Grassmann coordinate [7]:

$$S_{eff} = \int dt d\theta \, g_{ab}(\Phi) \dot{\Phi}^a D\Phi^b. \tag{2.10}$$

The super-covariant derivative, D, is given by $D = -i\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$. This sigma model possesses more supersymmetries than are apparent from this superspace formalism since the moduli space is hyperkähler, but we will not need to make those additional transformations explicit. Quantization of such a quantum mechanical model is a well-studied problem. Supersymmetric ground states correspond to zero modes of the Dirac operator on the moduli space, or equivalently to anti-holomorphic closed forms on M_k since the space is Calabi-Yau [8][9]. We can now move easily to the case with matter multiplets.

2.4. N=2 Yang-Mills Coupled to Matter

The Lagrangian for an N=2 hypermultiplet contains two chiral N=1 superfields M and $\widetilde{M}$ in conjugate representations of the gauge group. With bare masses set to zero, the matter Lagrangian is given by:

$$L_M = \int d^4\theta \left(M^\dagger e^{2V} M + \widetilde{M}^\dagger e^{-2V} \widetilde{M} \right) + \left(\sqrt{2} \int d^2\theta \widetilde{M} \Phi M + c.c. \right). \tag{2.11}$$

The components of the hypermultiplet are an $SU(2)_I$ doublet of scalar fields $\left(m, \widetilde{m}^\dagger\right)$ together with complex Weyl fermions $\left(\lambda, \widetilde{\lambda}\right)$ or equivalently two Majorana fermions. We shall discuss the case where M is in the fundamental representation of $SU(2)$ with hermitian generators T^l. In the Coulomb phase, the potential energy has no additional flat directions. Further, there are no zero modes for the scalars in the monopole background since the operator

$$-D^i D_i + (\sqrt{2}\phi^l T^l)^2,$$

is positive. However, the fermions do possess zero modes which form a bundle over the moduli space M_k. The bundle has dimension k and transition functions in $O(k)$. For each Majorana fermion zero mode λ_{on}, we introduce a Grassmann collective coordinate ψ^n, $n = 1, ..., k$. As usual, the ψ^n are dynamical and vary with time. The fermionic kinetic term for each Majorana fermions in the Lagrangian (2.11) provides a natural bundle metric $h_{nm}(\varphi) = \int d^3x\, \lambda_{0n}(\vec{x}, \varphi)\gamma^0 \lambda_{0m}(\vec{x}, \varphi)$ for the low-energy theory. The other ingredient needed to fully describe the effective action is a connection for the bundle. The connection also follows from the full action where we recall that A_0 can now contain terms with two fermions as well as the terms of order one in time derivatives. Let $\widetilde{D}_0$ be the restriction of D_0 obtained by setting the fermions (γ, ψ) to zero. An expression for the connection Ω then follows from the term

$$\begin{aligned} \int d^3x\, \lambda_{0n}\gamma^0 \widetilde{D}_0 \lambda_{0m} &= \int \lambda_{on}\gamma^0 \dot{\varphi}^a \left(\partial_a + \omega_a^l T^l\right) \lambda_{0m} \\ &= \dot{\varphi}^a \Omega_{anm}(\varphi) \end{aligned} \tag{2.12}$$

where ω_a satisfy (2.6). The effective action can then be described using fermionic superfields $\Psi^n = \psi^n + \theta F^n$ where the second component is an auxillary field, and

$$S_{eff} = \int dt d\theta\, g_{ab}(\Phi) \dot{\Phi}^a D\Phi^b + i h_{nm}(\Phi) \Psi^{n\alpha} (D\Psi^m_\alpha + D\Phi^a \Omega^m_{ap} \Psi^p_\alpha). \tag{2.13}$$

The additional index α affixed to the fermionic superfields runs from $1, \ldots, 2N_f$ since there are $2N_f$ Majorana fermions in the fundamental of an $SO(2N_f)$ flavor symmetry described in [3]. The symmetry properties of the bound states under this flavor group will be discussed further in section 4. The supersymmetry generator for this theory corresponds to the Dirac operator coupled to the $O(k)$ bundle connection. Supersymmetric ground states then correspond to normalizable zero modes of this twisted Dirac operator.

The case that we will focus on for the remainder of the paper is magnetic charge $k = 2$. Since the connection has an abelian component in this case, states in the Hilbert space of this theory are labelled by their $U(1)$ charge. Charge conjugation symmetry pairs a state with $U(1)$ charge n to one with charge -n. In the case of N_f hypermultiplets, the spectrum of $U(1)$ charges takes the form

$$|n>, |n+1>, \ldots, |n+2N_f> \tag{2.14}$$

where n is the $U(1)$ charge of the Fock vacuum. Charge conjugation then implies that $n = -N_f$. To summarize: the task of finding BPS bound states at magnetic charge two is equivalent to that of finding normalizable zero modes of the twisted Dirac operator acting on states with $U(1)$ charge $-N_f, \ldots, N_f$.

3. An Index Theorem for the Atiyah-Hitchin Manifold

To count the number of normalizable zero modes of the twisted Dirac operator, we will employ index theory. Clearly, we need to understand the two-monopole moduli space, its metric, the bundle and its connection. Fortunately, these topics have been well-studied, and most of the structures we require are known. We will benefit greatly from the work of [10], [11] and [12] in this analysis. However, the index computation is subtle since the moduli space is non-compact. To count the number of L^2 modes will require a careful analysis of boundary effects. First however, we must describe the geometry of the moduli space.

3.1. Monopole Moduli Space

As discussed in section 2.1, the BPS equations (2.4) for magnetic charge k have families of solutions with $4k$ parameters. For $k = 1$, these parameters describe translations and "large gauge transformations" of the standard BPS monopole solution. The moduli space in this case is just $\mathbf{R}^3 \times S^1$. For general k the space has the form

$$M_k = \mathbf{R^3} \times \frac{S^1 \times M_k^0}{\mathbf{Z}_k}, \tag{3.1}$$

where M_k^0 is a $4k-4$ real dimensional hyperkähler manifold equipped with a hyperkähler metric [10]. It is often useful to consider the $k-$fold cover $\widetilde{M}_k = \mathbf{R}^3 \times S^1 \times M_k^0$. The metric on $\widetilde{M}_k$ is flat in the $\mathbf{R}^3 \times S^1$ factors, so we will focus our discussion on M_k^0 – specifically, M_2^0.

Invariance of the BPS equations under the Euclidean group amounts to, in part, an $SO(3)$ group of isometries on M_2^0, the generic orbit of which is three dimensional. We identify $so(3)$ with the left invariant vector fields on $SO(3)$ in the usual manner. Let $\{X_1, X_2, X_3\}$ be a basis for $so(3)$ satisfying

$$[X_1, X_2] = -X_3, \;\; [X_3, X_1] = -X_2, \;\; [X_2, X_3] = -X_1,$$

and let $\{\sigma_i\}$ denote the dual frame. They satisfy the relation $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$.

In this notation, the metric on M_2^0 is constrained to be of the form

$$ds^2 = f(r)^2 dr^2 + a_1(r)^2\sigma_1^2 + a_2(r)^2\sigma_2^2 + a_3(r)^2\sigma_3^2. \tag{3.2}$$

Where convenient, the σ_i will be described by $SO(3)$ Euler angles $0 \leq \theta < \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 2\pi$:

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\ \sigma_3 &= d\psi + \cos\theta d\phi. \end{aligned}$$

M_2^0 also has the identification

$$(r, \theta, \phi, \psi) \equiv (r, \pi - \theta, \phi + \pi, -\psi). \tag{3.3}$$

Anti-self-duality of the curvature – following from hyperkählerity – tells us that

$$\frac{2a_2a_3}{f}\frac{da_1}{dr} = (a_2 - a_3)^2 - a_1^2,$$

as well as the equations obtained by cyclically permuting a_1, a_2, a_3.

By redefining the radial coordinate if necessary, we can take $f = -a_2/r$, and the range of r is then from π to ∞. For this choice the large r dependence of the metric is found to be

$$a_1 \approx a_2 \approx r\sqrt{1 - \frac{2}{r}}$$

$$a_3 \approx -\frac{2}{\sqrt{1-(2/r)}}$$

up to terms suppressed by e^{-r} [11]. Near the bolt coordinate singularity at $r = \pi$, we have

$$\begin{aligned} a_1 &\approx 2(r-\pi) \\ a_2 &\approx \frac{1}{2}(r-\pi)+\pi \\ a_3 &\approx \frac{1}{2}(r-\pi)-\pi. \end{aligned}$$

As discussed in the previous section, the zero energy solutions to the Dirac equation form an $O(k)$ bundle over the moduli space M_k. The bundle is trivial over the $\mathbf{R^3}$ factor in the decomposition of the moduli space given by (3.1). Further, Hitchin has shown that the connection on this bundle has anti-self-dual curvature.[2] Let us first consider the case $k = 1$ where the index bundle is an $O(1)$ bundle. While obviously flat, the bundle is not trivial, and is once twisted over the S^1 factor. We now focus our attention on the nontrivial case $k = 2$. Let Ind_2 denote the restriction of the bundle to $(S^1 \times M_2^0)/\mathbf{Z}_2$. The obstruction to orienting the bundle arises from the explicit discrete $\mathbf{Z}_2$ identification in (3.1), which we call I_3. In coordinates, I_3 is the identification

$$(\chi; r, \theta, \phi, \psi) \equiv (\chi+\pi; r, \theta, \phi, \psi+\pi), \tag{3.4}$$

where $0 \leq \chi < 2\pi$ is a coordinate for S^1. We can therefore pull back this bundle to $S^1 \times M_2^0$ and make a choice of orientation giving a $U(1)$ bundle $\widetilde{\mathrm{Ind}_2}$.

The anti-self-duality of the bundle curvature, and its invariance under the $SO(3)$ subgroup of the Euclidean group of symmetries of the BPS equations, fix the curvature to lie entirely in M_2^0 and have the form

$$\Omega = d\alpha \wedge \sigma_1 + \alpha\sigma_2 \wedge \sigma_3. \tag{3.5}$$

The function $\alpha(r)$ falls off as $e^{-r/2}$ as $r \to \infty$, and has a normalization $\alpha(\pi) = \pm\frac{1}{2}$ [12]. The sign ambiguity depends on the choice of orientation, and will play no role in the following analysis.

[2] As referred to in [12].

3.2. The Index Formula

In this section, we consider the restriction of $\widetilde{\mathrm{Ind}}_2$ to $(p, M_2^0) \cong M_2^0$, where p is any point on S^1.[3] I inherits a connection from $\widetilde{\mathrm{Ind}}_2$. We wish to compute the dimension of the kernel of the Dirac operator D on spinors with values in I^n. First we need a vanishing theorem.

Proposition 3.2.1 *Let M be an infinite volume spin four-manifold with zero scalar curvature. Let E be a line bundle over M whose curvature r^E is an anti-self-dual two-form. Then the kernel of the Dirac operator D_E^- on $S^- \otimes E$ is zero.*
Proof. Integrating by parts and using the Bochner-Lichnerowicz formula ([13], Theorem 8.17) we have

$$\|D_E^- f\|^2 = \|\nabla f\|^2 + (R^E f, f),$$

where $R^E = \sum_{i<j} e_i e_j r^E_{i,j}$, and e_i denotes Clifford multiplcation by the i^{th} vector in a frame. In an oriented frame, we may write the projection onto $S^- \otimes E$ as $(1 - e_0 e_1 e_2 e_3)/2$. The anti-self-duality of r^E implies

$$\sum_{i<j} e_i e_j r^E_{i,j} (1 - e_0 e_1 e_2 e_3)/2 = 0.$$

Hence,

$$\|D_E^- f\|^2 = \|\nabla f\|^2,$$

for f a section of $S^- \otimes E$. An element of the kernel of D_E^- is therefore covariantly constant. This implies its norm is covariantly constant. On an infinite volume manifold the only covariantly constant square integrable section is zero. □

Corollary 3.2.2 *The L^2 index of the Dirac operator*

$$D^+ : L^2(M_2^0, S^+ \otimes I^n) \to L^2(M_2^0, S^- \otimes I^n)$$

is the dimension of the kernel of D^+.

Thus we are left with the computation of the L^2 index. We recall the modifications required to compute the index on a complete noncompact manifold. Let e^{-sD^2} denote the heat operator associated to D^2. As $s \to \infty$, e^{-sD^2} converges weakly to projection, Π, onto

[3] Though $\widetilde{\mathrm{Ind}}_2$ wraps nontrivially around the S^1, there is no ambiguity in defining I, up to isomorphism.

the kernel of D. Hence, the integral of $tre^{-sD^2}(x,x)$ over a compact subset C converges to $\int_C tr\Pi(x,x)dx$, and $\int_C tr\tau e^{-sD^2}(x,x)dx \to \int_C tr\tau\Pi(x,x)dx$, where τ denotes Clifford multiplication by the volume element. $\tau = e_0e_1e_2e_3$ in the notation of the preceding proposition. The L^2 index of D is given by the integral

$$\mathrm{Ind}(D) = \int_{M_2^0} tr\tau\Pi(x,x)dx = \int_{M_2^0} lim_{s\to\infty} tr\tau e^{-sD^2}(x,x)dx.$$

On a compact manifold, Y, with Dirac operator D_Y, $\int_Y tr\tau e^{-sD_Y^2}(x,x)dx$ is independent of s; hence, in the above discussion the $s \to \infty$ can be replaced by $s \to 0$, and one obtains

$$\mathrm{Ind}(D_Y) = lim_{s\to 0} \int_Y tr\tau e^{-sD_Y^2}(x,x)dx.$$

In the latter limit, the traces are easily computed. In general $\frac{d}{ds} tr\tau e^{-sD^2}(x,x)$ can be expressed as the divergence of some vector field $V(s)$, with

$$\int_C divV(s)dx = \int_{\partial C} tre_0\tau De^{-sD^2} d\sigma.$$

Here e_0 is Clifford multiplication by the unit outward normal to ∂C and $d\sigma$ is the induced volume form on ∂C. (See [6].) This explains the s independence of the trace in the compact case and yields the following expression for the index on M_2^0 (see, e.g. [14], [15], and [6]):

$$\mathrm{Ind}(D) = lim_{L\to\infty}(lim_{s\to 0} \int_{r<L} \mathrm{tr}\tau e^{-sD^2} dx + \int_0^\infty \int_{r=L} tre_0\tau De^{-sD^2} d\sigma ds).$$

The Atiyah-Singer index theorem gives

$$lim_{L\to\infty}(lim_{s\to 0} \int_{r<L} \mathrm{tr}\tau e^{-sD^2} dx) =$$
$$\frac{1}{24\cdot 8\pi^2} \int_{M_2^0} \mathrm{Tr}(R\wedge R) + \frac{1}{8\pi^2} \int_{M_2^0} \mathrm{Tr}(\Omega\wedge\Omega).$$

We write

$$\mathrm{Ind}(D) = \frac{1}{24\cdot 8\pi^2} \int_{M_2^0} \mathrm{Tr}(R\wedge R) + \frac{1}{8\pi^2} \int_{M_2^0} \mathrm{Tr}(\Omega\wedge\Omega) + \delta_D, \tag{3.6}$$

where

$$\delta_D = lim_{L\to\infty} \int_0^\infty \int_{r=L} tre_0\tau De^{-sD^2} d\sigma ds.$$

From the form of the metric (3.2), one computes the curvature in the orthonormal frame

$$\{\nu_0, \nu_1, \nu_2, \nu_3\} \equiv \{f\, dr, a_1\sigma_1, a_2\sigma_2, a_3\sigma_3\}$$

to be

$$\begin{array}{lllll} R_{10} & = & R_{23} & = & \frac{1}{a_1 f}(\frac{\partial}{\partial r}A)\nu_0 \wedge \nu_1 + \frac{1}{a_2 a_3}(A+B+C-1-2BC)\nu_2 \wedge \nu_3, \\ R_{20} & = & R_{31} & = & \frac{1}{a_2 f}(\frac{\partial}{\partial r}B)\nu_0 \wedge \nu_2 + \frac{1}{a_1 a_3}(A+B+C-1-2AC)\nu_3 \wedge \nu_1, \\ R_{30} & = & R_{12} & = & \frac{1}{a_3 f}(\frac{\partial}{\partial r}C)\nu_0 \wedge \nu_3 + \frac{1}{a_1 a_2}(A+B+C-1-2AB)\nu_1 \wedge \nu_2, \end{array}$$

where

$$A = (a_2^2 + a_3^2 - a_1^2)/2a_2a_3,$$

$$B = (a_3^2 + a_1^2 - a_2^2)/2a_3a_1,$$

$$C = (a_1^2 + a_2^2 - a_3^2)/2a_1a_2.$$

Anti-self-duality holds for the choice of orientation $\nu_0 \wedge \nu_1 \wedge \nu_2 \wedge \nu_3$, and we see that

$$\mathrm{Tr}(R \wedge R) = d\left[\left(-4(A+B+C-1)^2 + 16ABC\right)\sigma_1 \wedge \sigma_2 \wedge \sigma_3\right],$$

which by Stokes' theorem (note that the boundary orientation induced here is $-\nu_1 \wedge \nu_2 \wedge \nu_3$) and the asymptotic formulas for a_1, a_2, and a_3 gives

$$\frac{1}{24 \cdot 8\pi^2}\int_{M_2^0} \mathrm{Tr}(R \wedge R) = -\frac{1}{6}.$$

Note that we have divided by a factor of two due to the $\mathbf{Z}_2$ identification (3.3).

The next term in the bulk contribution is also simple to compute, since we have, from (3.5),

$$\Omega \wedge \Omega = n^2 d(\alpha^2 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3)$$

for the bundle I^n. Using $\alpha^2(\infty) = 0$ and $\alpha^2(\pi) = 1/4$, we find

$$\frac{1}{8\pi^2}\int_{M_2^0} \mathrm{Tr}(\Omega \wedge \Omega) = \frac{n^2}{8}.$$

Combining the terms, we arrive at

$$\mathrm{Ind}(D) = n^2/8 - 1/6 + \delta_D. \tag{3.7}$$

We now turn to the calculation of the boundary contribution δ_D.

3.3. An Equivalent Index Problem

For K some large positive constant, equip $(K,\infty)\times SO(3)$ with the metric $f(r)^2dr^2 + \sum a_i^2(r)\sigma_i^2$. This metric descends to a metric on $(K,\infty)\times SO(3)/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ action is the one generated by (3.3). This is just the Atiyah-Hitchin metric described in (3.2) near ∞.

In order to reduce the computation of the index of the Dirac operator to computations essentially the same as those carried out in [14], we will first make a conformal change in the metric of the moduli space in a neighborhood of ∞. This change is not essential to computing the defect δ_D but simplifies the estimation of error terms involved in constructing approximations to resolvents and heat operators. By the conformal invariance of the Pontrjagin classes, we know that such a conformal change will not change the value of the bulk contribution. Hence, in order to justify it, we need only check that it preserves the index. First let us specify a new metric which, for large r, is given by

$$\begin{aligned} g &= \frac{f(r)^2}{a_2^2}dr^2 + \sum_1^3 \frac{a_i^2}{a_2^2}\sigma_i^2 \\ &= \frac{dr^2}{r^2} + (1+A_1)\sigma_1^2 + \sigma_2^2 + (\frac{4}{r^2(1-2/r)^2} + A_3)\sigma_3^2, \end{aligned} \tag{3.8}$$

with A_j doubly exponentially decreasing functions of t (i.e. $O(e^{kt}e^{-e^t})$). Set $t = ln(r)$. The metric is then

$$dt^2 + (1+A_1)\sigma_1^2 + \sigma_2^2 + (\frac{4}{e^{2t}(1-2e^{-t})^2} + A_3)\sigma_3^2.$$

We fix an orthonormal frame $\{Y_i\}$ on the $\mathbf{Z}_2$ cover of our space with $Y_0 = \frac{\partial}{\partial t}$, $Y_1 = (1+A_4)X_1$, $Y_2 = X_2$, and $Y_3 = (e^t/2 - 1 + A_5)X_3$, with A_4, A_5 doubly exponentially decreasing. Of course this frame does not descend to one defined globally on the $\mathbf{Z}_2$ quotient, but that will not affect our computations. Quantities involving the squares of these vector fields will descend to the quotient.

Neglecting doubly exponentially decreasing terms, we have the commutation relations:

$$\begin{aligned} [Y_1, Y_2] &\sim -\frac{2e^{-t}}{1-2e^{-t}}Y_3, \\ [Y_3, Y_1] &\sim -e^t(1/2 - e^{-t})Y_2, \\ [Y_2, Y_3] &\sim -e^t(1/2 - e^{-t})Y_1, \\ [Y_0, Y_3] &\sim (1-2e^{-t})^{-1}Y_3. \end{aligned}$$

From these relations, we can check modulo doubly exponentially decreasing terms that

$$
\begin{aligned}
(\nabla_{Y_3}Y_1, Y_2) &= -(e^t/2 - 1 - (e^t/2-1)^{-1}/2),\\
(\nabla_{Y_1}Y_2, Y_3) &= -(e^t - 2)^{-1},\\
(\nabla_{Y_2}Y_3, Y_1) &= (e^t - 2)^{-1},\\
(\nabla_{Y_3}Y_0, Y_3) &= -(1 - 2e^{-t})^{-1}.
\end{aligned}
$$

For the remainder of this section, let D denote the Dirac operator with respect to this new metric and denote by $\widetilde{D}$ the Dirac operator associated to the old metric $\tilde{g} = e^{2loga_2}g$. We recall (see [13]) that there is a local isometry ϕ between the spaces of spinors determined by the two conformal structures such that

$$\widetilde{D}s = e^{-3loga_2/2}D(e^{3loga_2/2}\phi(s)).$$

Hence, the map

$$\Phi : s \to e^{3loga_2/2}\phi(s)$$

takes harmonic spinors to harmonic spinors. This would induce an isomorphism between spaces of L^2 harmonic spinors if Φ preserved the L^2 condition.

Proposition 3.3.1 *The map Φ is an isomorphism between spaces of L^2 harmonic spinors.*
Proof. Let $h \in L^2(\widetilde{g})$. Then

$$\infty > \int |h|^2 dv_{\tilde{g}} \sim \int |h|^2 e^{4t} dv_g.$$

Hence, $\int |e^{3t/2}\phi(h)|^2 e^t dv_g < \infty$, and Φ maps L^2 harmonic spinors to L^2 harmonic spinors. To obtain an isomorphism, we must also show that the inverse map also preserves the L^2 condition. It suffices to show that for $H \in \mathrm{Ker}(D)$, $e^{t(1/2+a)}H \in L^2(g)$ for some positive a. We shall prove this estimate below. First we need a preliminary discussion of the holonomy of the index bundle, I^n.

The fundamental group of $SO(3)/\mathbf{Z}_2$ is generated by the inclusion of the circle K obtained by exponentiating the Y_3 vector. This is the fiber of the fibration

$$\{L\} \times SO(3)/\mathbf{Z}_2 \to \{L\} \times SO(3)/K\mathbf{Z}_2. \tag{3.9}$$

According to [12], the holonomy about this circle fiber is given by $e^{\pm i\pi n/2}$ when $L = \infty$. They show this by integrating the curvature over the Atiyah-Hitchin cone [10]. Hence, for

finite L, the holonomy differs from this factor by $e^{i\pi\epsilon(L)}$, where ϵ is doubly exponentially decreasing. This can be seen by integrating the doubly exponentially decreasing curvature over the subset of the cone with $r > L$. Note that the $\mathbf{Z}_2$ action prevents us from globally fixing the sign since Y_3 is only globally defined up to a factor of ± 1. We will assume Y_3 chosen so that the holonomy $\sim e^{-i\pi n/2+i\epsilon(L)}$.

It is more convenient to work with periodic frames than with covariantly constant ones. Multiplying a Y_3-covariantly constant frame for I^n by $e^{-i\psi(n/4-\epsilon/2)}$ gives a connection form of $\frac{in}{4}Y_3^*$ modulo a doubly exponentially decreasing term.

Taking now a frame for the spinors determined by our (periodic) frame $\{Y_0, Y_1, Y_2, Y_3\}$ and the periodic frame for I^n, we can Fourier expand sections in the Y_3 direction. This can even be done globally, although the Fourier coefficients then become sections of a $\mathbf{Z}_2$ quotient of powers of the Hopf bundle over S^2.

Setting $T = Y_0$, we can write

$$D^2 = -T^2 - \nabla^2_{Y_1} - \nabla^2_{Y_2} - \nabla^2_{Y_3} + \nabla_{\nabla_{Y_3} Y_3},$$

modulo rapidly decreasing terms. Then taking the k^{th} Fourier component with respect to the circle action associated to Y_3, we have

$$\begin{aligned} D^2 = & - T^2 + T - (e^t/2-1)^2(ik + in/4 - 1/2(1 - [e^t/2-1]^{-2}/2)e_1e_2 \\ & - \frac{e^t}{4(e^t/2-1)^2}e_0e_3)^2 \end{aligned}$$

plus a positive operator and rapidly decreasing terms. This operator is conjugate to

$$\begin{aligned} & - T^2 + 1/4 - (e^t/2-1)^2(ik + in/4 - 1/2(1 - [e^t/2-1]^{-2}/2)e_1e_2 \\ & - \frac{e^t}{4(e^t/2-1)^2}e_0e_3)^2 \end{aligned}$$

plus a positive operator and rapidly decreasing terms. The smallest eigenvalue of $-(ik + in/4 - 1/2e_1e_2)^2$ is $1/16$ when $n \not\equiv 2 \pmod 4$, since k is an integer. Hence, a standard maximum principle argument (or differential inequality) says that for $n \not\equiv 2 \pmod 4$, any L^2 harmonic spinor is doubly exponentially decreasing and therefore maps to an $L^2(\tilde{g})$ harmonic spinor under the previously described map. Moreover, this estimate implies that D is Fredholm with no continuous spectrum in the case of restricted n. We are left to prove the proposition in the case $n \equiv 2 \pmod 4$. In this case, $(ik + in/4 - 1/2e_1e_2)$ can have a kernel. On this kernel, the $-\nabla_3^2$ term is dominated by $-(\frac{e^t}{4(e^t/2-1)^2}e_0e_3)^2 = 1/4$ modulo

exponentially decreasing terms. Hence, the decay of the component corresponding to the kernel of $(ik+in/4-1/2e_1e_2)$ is, by a maximum principle argument, of the order $O(e^{-t/2})$. As the volume of the $SO(3)$ orbit at t is $O(e^{-t})$, $e^{t(1/2+a)}H \in L^2(g)$ for $H \in \mathrm{Ker}(D)$. This implies that Φ^{-1} takes L^2 harmonic spinors to L^2 harmonic spinors, completing the proof of our proposition. $\square$

This proof shows that the essential spectrum of D^2 is bounded away from zero. Hence, D is Fredholm. A similar argument works to show when $n \not\equiv 2 \pmod 4$ that $\widetilde{D}$ is Fredholm.

3.4. Parametrices

For the convenience of the reader, we gather in this section aspects of the construction of the approximate heat kernels. The idea here is to determine an operator which is an appropriate approximation to e^{-sD^2} for use in computing δ_D. We will define such an operator and find that only a few relevant components contribute to our calculation.

Using the functional calculus, we write

$$e^{-sD^2} = \frac{1}{2\pi i}\int_\gamma e^{-s\lambda}(D^2-\lambda)^{-1}d\lambda,$$

where $\gamma \subset C$ is a simple curve surrounding the spectrum of D^2. Thus, it suffices to approximate $(D^2-\lambda)^{-1}$.

We construct the semilocal approximation to $(D^2-\lambda)^{-1}$ inductively using a continuous Fourier transform in the base variables (in a coordinate neighborhood) and a discrete Fourier expansion in the fibers. More precisely, we consider contractible open subsets of RP^2 (the base of the fibration) over which the fibration is trivial. As in section 3.3, we Fourier expand the sections of the twisted spinor bundles. Fix a coordinate neighborhood V on the base space of the fibration. On the k^{th} Fourier component D^2 has the form

$$\begin{aligned}&-T^2+T-\nabla^2_{Y_1}-\nabla^2_{Y_2}-(e^t/2-1)^2(ik+in/4-1/2(1-[e^t/2-1]^{-2}/2)e_1e_2\\&-\frac{e^t}{4(e^t/2-1)^2}e_0e_3)^2\end{aligned}$$

plus rapidly decreasing terms.

Let

$$K^2 = (e^t/2-1)^2(ik+in/4-1/2(1-[e^t/2-1]^{-2}/2)e_1e_2)^2.$$

Let $\|v\|^2 = \sum g^{ij}v_{ij}$. In a nice frame, let $f=\sum_k f_k(x)e^{ik\psi}$ denote the Fourier expansion of a section f, and let $\hat{f}_k(v)$ denote the Fourier transform of f_k. Then the action of D^2 on f is

given via the inverse Fourier transform by operating on $\hat{f}_k$ by multiplication by $\|v\|^2 + K^2$ plus some lower order operator L, which is readily computed. Hence, to invert $(D^2 - \lambda)$ approximately, it suffices to invert $(\|v\|^2 + K^2 - \lambda + L)$ approximately. We construct such an inverse in the form $\sum_{l=0}^{N} \frac{h_l}{(\|2\pi v\|^2 + K^2 - \lambda)^{l+1}}$. For the generic case, K is very large and so high powers of $(\|2\pi v\|^2 + K^2 - \lambda)^{-1}$ are rapidly decreasing. Hence, the numerators are constructed by an inductive process, so that when acted on by $D^2 - \lambda$ (in the guise of $\|v\|^2 + K^2 + L$) we obtain 1 plus a high power of $(\|2\pi v\|^2 + K^2 - \lambda)^{-1}$. The inductive construction is given as follows. Set

$$h_0(x, x', \lambda, v, k) = \text{Identity}.$$

Write

$$\begin{aligned}
&(D^2 - \lambda)[(2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} (\|2\pi v\|^2 + K^2 - \lambda)^{-l-1} h_l(x, x', \lambda, v, k)\\
= &(2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} (\|2\pi v\|^2 + K^2 - \lambda)^{-l} h_l(x, x', \lambda, v, k) -\\
&2\nabla((2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v}) \cdot \nabla\{(\|2\pi v\|^2 + K^2 - \lambda)^{-l-1} h_l(x, x', \lambda, v, k)\}\\
&+ (2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} \Delta_2\{(\|2\pi v\|^2 + K^2 - \lambda)^{-l-1} h_l(x, x', \lambda, v, k)\}\\
= &(2\pi)^{-1} e^{ik(\psi-\psi')} e^{i2\pi(x-x')\cdot v} (\|2\pi v\|^2 + K^2 - \lambda)^{-l} [h_l(x, x', \lambda, v, k) + R_l],
\end{aligned}$$

where $\Delta_2 = D^2 - K^2$. Set

$$h_{l+1} = -(R_l) e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} (\|2\pi v\|^2 + K^2 - \lambda)^{l+1}.$$

Then one obtains formally

$$(D^2 - \lambda) \sum_k \sum_{l=0}^{N} \int_{\mathbf{R}^3} (2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} \frac{h_l(x, x', \lambda, v, k) dv}{(\|2\pi v\|^2 + K^2 - \lambda)^{l+1}} =$$

$$\text{Identity} + \int_{\mathbf{R}^3} (R_N) dv,$$

and we take for some large N,

$$\sum_k \sum_{l=0}^{N} \int_{\mathbf{R}^3} (2\pi)^{-1} e^{ik\cdot(\psi-\psi')} e^{i2\pi(x-x')\cdot v} \frac{h_l(x, x', \lambda, v, k) dv}{(\|2\pi v\|^2 + K^2 - \lambda)^{l+1}}$$

for our semilocal approximation to $(D^2 - \lambda)^{-1}$ (pre- and post-multiplied by cutoff functions in the usual way and summed over a cover, etc.).

Write

$$Dh_l(x,x,\lambda,v,k) = \sum_\sigma (\|2\pi v\|^2 + K^2 - \lambda)^{-\sigma} h_{l,\sigma}(x,v,k),$$

and

$$h_{l,\sigma}(x,v,k) = \sum_{0\le|A|+B\le 2\sigma+1, C\ge 0} h_{l,\sigma,A,B,C}(x) v^A (e^t k)^B e^{-tC},$$

with $h_{l,\sigma,A,B,C}(x)$ bounded.

In the following section, we shall study traces of the operator

$$\int_0^\infty \int_\gamma e^{-s\lambda} \int_{\mathbf{R}^3} \frac{h_{l,\sigma,A,B,C}(x) v^A (e^t k)^B e^{-tC}) dv d\lambda ds}{(\|2\pi v\|^2 + K^2 - \lambda)^{l+1+\sigma}} \le$$

$$O((ke^t)^{[B+|A|-2l-2\sigma-C+1]}) \le O((ke^t)^{[2-2l-C]}.$$

Hence, we see without yet using the trace identities that only the terms with $l = 0$ or 1 can contribute to the trace. Let us now recall the basic trace lemma.

Lemma 3.4.1 *For $i_1, \cdots, i_j$ distinct and $j > 0$,*

$$tr e_{i_1} \cdots e_{i_j} = 0.$$

So in computing the trace of $e_0 \tau Dh_l(x,x,\lambda,v,k)$ we get a nonzero contribution only from those terms which introduce enough Clifford factors to cancel $e_0\tau = -e_1e_2e_3$ – and no more. No term in Dh contributes a factor of e_3 without also contributing an extra e_0 except the term

$$e_3(e^t/2 - 1)(ik + in/4 - 1/2(1 - [e^t/2 - 1]^{-2}/2)e_1e_2)h$$

so this is the only term which can have nonzero trace. We will compute the contribution of Dh_0 to this trace. A similar computation shows that the contribution of Dh_1 is rapidly decreasing.

We compute

$$\begin{aligned}
&\frac{1}{2\pi i}\int_\gamma e^{-t\lambda} \sum_k \int_{\mathbf{R}^3} (2\pi)^{-1} e^{ik(\psi-\psi')} e^{i2\pi(x-x')\cdot v} \frac{h_0(x,x',\lambda,v,k) dv d\lambda}{(\|2\pi v\|^2 + K^2 - \lambda)} \\
=&\frac{1}{2\pi i}\int_\gamma e^{-t\lambda} \sum_k \int_{\mathbf{R}^3} (2\pi)^{-1} e^{ik(\psi-\psi')} e^{i2\pi(x-x')\cdot v} \frac{dv d\lambda}{(\|2\pi v\|^2 + K^2 - \lambda)} \\
=&\sum_k (4\pi t)^{-3/2} e^{-|x-x'|^2} e^{-tK^2} (2\pi)^{-1} e^{ik(\psi-\psi')}.
\end{aligned}$$

Recall we need to compute the trace of

$$\sum_k e_0\tau e_3(e^t/2-1)(ik+in/4-1/2(1-[e^t/2-1]^{-2}/2)e_1e_2)(4\pi t)^{-3/2}\times$$
$$e^{-|x-x'|^2/4t}e^{-tK^2}(2\pi)^{-1}e^{ik(\psi-\psi')}$$

along $x=x'$, $\psi=\psi'$ where we recall that τ is the volume form $e_0e_1e_2e_3$. Integrating this trace over the S^1 fiber reduces us to computing

$$\sum_k tre_1e_2(e^t/2-1)(ik+in/4-1/2(1-[e^t/2-1]^{-2}/2)e_1e_2)(4\pi t)^{-3/2}e^{-tK^2}. \qquad (3.10)$$

3.5. Defect Computations

We can now turn to the computation of δ_D defined in (3.7). Let e_i denote Clifford multiplication by Y_i. In our situation, we have the following L^2 index theorem as we have previously discussed.

Proposition 3.5.1 *The index on I^n is given by the expression*

$$\mathrm{Ind}(D)=n^2/8-1/6+lim_{L\to\infty}\int_{t=L}\int_0^\infty tre_0\tau De^{-sD^2}ds.$$

The computation of δ_D is essentially the same as the index computation in [15] in the special case of a smooth divisor. Hence we will restrict ourselves here to providing an outline of the methods involved. The computation requires two steps: first we need to construct an explicit approximation E_s to e^{-sD^2} so that

$$lim_{L\to\infty}\int_{t=L}\int_0^\infty tre_0\tau D(e^{-sD^2}-E_s)ds=0.$$

Using the results of the previous section, E_s takes the form (suppressing patching and partitions of unity etc.) :

$$E_s(x,y)=(2\pi i)^{-1}\int_\gamma e^{-t\lambda}\sum_k\sum_{l=0}^N\int_{R^3}(2\pi)^{-1}e^{ik(\psi-\psi')}e^{i2\pi(x-x')\cdot v}\times$$
$$\frac{h_l(x,x',\lambda,v,k)dv}{(\|2\pi v\|^2+K^2-\lambda)^{l+1}}.$$

Second, we need to compute

$$lim_{L\to\infty}\int_{t=L}\int_0^\infty tre_0\tau DE_sds.$$

For N large and $n \not\equiv 2 \pmod 4$, the presence of the product of $-(e^t/2-1)^2$ and the square of the Fourier coefficients in the exponent can be used to show that $D(E_s - e^{-sD^2})$ has very small trace norm when multiplied by a cutoff function supported near ∞.

When $n \equiv 2 \pmod 4$, this construction does not yield good error estimates on the kernel of $(ik + in/4 - 1/2e_1e_2)$. On this subspace, we must use a parametrix construction which is global on the entire $SO(3)$ orbit. This is similar to the construction of the parametrix for singular m in [14] – see section 6 in particular. There is a natural limiting Dirac operator as $L \to \infty$ on this subspace, and we construct our heat operator (restricted to this subspace) as a perturbation of the heat operator associated to this Dirac operator. We need not compute this operator explicitly. It suffices to note that it does not contribute any Clifford multiplication factors to prevent cancellation in the super trace. The necessary Clifford factors arise in the perturbation expansion, but with $O(e^{-L})$ coefficients. Hence this term contributes nothing to our trace. The trace over the kernel of $(ik+in/4-1/2e_1e_2)$ of the parametrix obtained by the semi-local construction also contributes zero. Hence, we may use the semi-local construction for our entire computation.

Set $U = (e^L/2-1)^{-1}$. Returning to the computation of the traces with the semi-local parametrix, we see from (3.10) that

$$\begin{aligned}\int_{t=L}\int_0^\infty tre_0\tau De^{-sD^2}ds = (4\pi)\sum_k \int_0^\infty & tre_1e_2U^{-1}(ik+in/4-(1/2- \\ & U^2/4)e_1e_2)e^{\frac{s}{U^2}(ik+in/4-(1/2-U^2/4)e_1e_2)^2}\times \\ & (4\pi s)^{-3/2}ds + O(e^{-L}).\end{aligned}$$

We simplify and apply the Poisson summation formula to write this as

$$
\begin{aligned}
&4\pi \int_0^\infty \sum_p \int_{\mathbf{R}} tre_1e_2U^{-1}(ix+in/4-(1/2-U^2/4)e_1e_2)\times \\
&e^{\frac{t}{U^2}(ix+in/4-(1/2-U^2/4)e_1e_2)^2}(4\pi t)^{-3/2}e^{-2\pi ipx}dxdt + O(e^{-L}) \\
=\,&4\pi \int_0^\infty \sum_p \int_{\mathbf{R}} tre_1e_2U^{-1}ixe^{\frac{-t}{U^2}x^2}(4\pi t)^{-3/2}e^{-2\pi ipx}\times \\
&e^{2\pi ip(n/4+i(1/2-U^2/4)e_1e_2)}dxdt + O(e^{-L}) \\
=\,&4\pi tr \int_0^\infty \sum_p e^{2\pi ip(n/4+i(1/2-U^2/4)e_1e_2)}e_1e_2iU^{-1}(-2\pi i)^{-1}\times \\
&\frac{\partial}{\partial p}e^{-\pi^2p^2U^2/t}\pi^{1/2}Ut^{-1/2}(4\pi t)^{-3/2}dt + O(e^{-L}) \\
=\,&tr\sum_{p\neq 0} e^{2\pi ip(n/4+i(1/2-U^2/4)e_1e_2)}e_1e_2iU^{-1}(-2\pi i)^{-1}\frac{\partial}{\partial p}\frac{1}{2\pi^2p^2U} + O(e^{-L}) \\
=\,&tr\sum_{p\neq 0} e^{2\pi ip(n/4+i(1/2-U^2/4)e_1e_2)}e_1e_2\frac{1}{2\pi^3p^3U^2} + O(e^{-L}).
\end{aligned}
$$

We now eliminate terms using the trace identities. Replacing the factor $e^{\pi p(U^2/2e_1e_2)}$ by $e^{\pi p(U^2/2e_1e_2)} - 1$ does not change the trace, as $tre_1e_21 = 0$. Moreover, we have that

$$e^{\pi p(U^2/2e_1e_2)} - 1 = \pi p(U^2/2e_1e_2) + O(e^{-4L}).$$

Hence, the defect reduces to

$$tr\sum_{p\neq 0} e^{\pi ipn/2}(-1)^p\pi pU^2e_1e_2e_1e_2\frac{1}{4\pi^3p^3U^2} + O(e^{-L}),$$

which finally gives,

$$\delta_D = \sum_{p\neq 0} e^{\pi ipn/2}(-1)^{p+1}\frac{1}{\pi^2p^2}.$$

We compute this for different values of n (mod 4). When $n=0$,

$$\delta_D = \zeta(2)/\pi^2,$$

where $\zeta(s)$ denotes the Riemann zeta function. As is well known, $\zeta(2) = \pi^2/6$. Hence, for $n = 0$ (mod 4)

$$\delta_D = 1/6.$$

If $n = 1$, (mod 4) we obtain

$$-\pi^{-2}\sum_{p\neq 0} p^{-2} e^{i\pi p 3/2} = -\pi^{-2}\sum_{p\neq 0} p^{-2}(-1)^p i^p = 1/2\sum_{p>0}\pi^{-2}p^{-2}(-1)^{p+1} = 1/24.$$

If $n = 2$, (mod 4) we obtain

$$-\pi^{-2}\sum_{p\neq 0} p^{-2} e^{i\pi p 2} = -1/3.$$

If $n = 3$, (mod 4) we obtain

$$-\pi^{-2}\sum_{p\neq 0} p^{-2} e^{i\pi p 5/2} = -\pi^{-2}\sum_{p\neq 0} p^{-2} i^p = -1/2\sum_{p>0}\pi^{-2}p^{-2}(-1)^{p} = 1/24.$$

Before proceeding, let us summarize the results of this index computation:

$$\text{dimKer}(D_{I^n}) = n^2/8 - 1/6 + \begin{cases} 1/6, & n \equiv 0 \bmod 4 \\ 1/24, & n \equiv 1 \\ -1/3, & n \equiv 2 \\ 1/24, & n \equiv 3 \end{cases}$$

3.6. Sections of Ind_2 *and Determining the Electric Charges*

Until now we have been working on the space M_2^0, although we are actually interested in sections of the index bundle Ind_2 over $(S^1 \times M_2^0)/I_3$, where I_3 is a $\mathbf{Z}_2$ involution. Since the bundle is trivial over the $\mathbf{R}^3$ portion of the monopole moduli space, we can restrict our attention to the dependence of the bound state wavefunction on $S^1 \times M_2^0$. To obtain the full wavefunctions, we can simply multiply by a factor of $e^{i\vec{p}\cdot\vec{x}}$, giving the particles momentum.

To define sections of the quotient bundle Ind_2, one must equivariantly lift the $\mathbf{Z}_2$ action from the space $S^1 \times M_2^0$ to the total space of the bundle $\widetilde{\mathrm{Ind}}_2$ over $S^1 \times M_2^0$. The M_2^0 piece of this bundle has been discussed in section 3.1. Since M_2^0 is homotopic to S^2, it is convenient to describe this piece of the bundle as a bundle over S^2 – it is in fact the Hopf bundle (the total bundle over M_2^0 is then isomorphic to the pullback) [12]. The Hopf bundle $S^3 \to S^2$ is the quotient map $\vec{z} \equiv (z_1, z_2) \mapsto [z_1, z_2]$, where $|z_1|^2 + |z_2|^2 = 1$ and $[z_1, z_2]$ is the point in $\mathbf{CP}^1 \cong S^2$ denoting the complex line through (z_1, z_2). The fiber is clearly $U(1)$. Then by the homotopy, we can think of the total space of the bundle Ind_2 over $S^1 \times M_2^0$ as a bundle P over $S^1 \times S^2$, defined as follows [12]:

$$P = \mathbf{R} \times S^3 / \left\{(t, \vec{z}) \sim (t + 2\pi, -\vec{z})\right\}. \tag{3.11}$$

The bundle map is trivial on the first coordinate and the Hopf map given above on $\vec{z}$. Note that on the base $t \sim t + 2\pi$, but we don't have $(t, \vec{z}) \sim (t + 2\pi, \vec{z})$ on the total space. Note, too, the nontrivial S^1 twist encoding the holonomy over the S^1 factor. The bundle over M_2^0 given by pulling back the Hopf bundle in the manner discussed above is just the bundle I of the previous sections. Now the relation (3.11) is equivalently

$$\widetilde{\mathrm{Ind}}_2 = \mathbf{R} \times I / \{(t, e) \sim (t + 2\pi, -e)\}, \tag{3.12}$$

where $-e$ is $-1 \cdot e$, with -1 acting along the fiber.

To get to Ind_2, we must quotient this bundle by $\mathbf{Z}_2$. This procedure involves lifting I_3 to $\widetilde{I}_3$ acting on the total space of $\widetilde{\mathrm{Ind}}_2$. By "lift," we mean a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{I}_3} & P \\ \downarrow & & \downarrow \\ S^1 \times S^2 & \xrightarrow{I_3} & S^1 \times S^2. \end{array}$$

In describing the lift, we will for convenience work with the bundle P defined in (3.11) and use the same notation $\widetilde{I}_3$ and I_3 – our statements can be "pulled back" to $\widetilde{\mathrm{Ind}}_2$. In terms of S^2, the I_3 on M_2^0 acts downstairs as the antipodal map on $S^2 : (z_1, z_2) \mapsto (-\overline{z}_2, \overline{z}_1)$. Then $\widetilde{I}_3$ maps

$$(t, (z_1, z_2)) \to (t + \pi, (-\overline{z}_2, \overline{z}_1)).$$

One easily checks that $\widetilde{I}_3$ squares to the identity as a result of the equivalence (3.11), and that the diagram is commutative. Now the quotient

$$\mathrm{Ind}_2 \equiv \widetilde{\mathrm{Ind}}_2 / \widetilde{I}_3$$

makes sense as a bundle, with a well-defined bundle map $\pi[(t, e)] = [(t, \pi(e))]$ because of the equivariance.

Recall how an automorphism acts on global sections: if $s : S^1 \times M_2^0 \to \widetilde{\mathrm{Ind}}_2$ is a global section, then

$$s \to \widetilde{I}_3^{\,-1} \circ s \circ I_3$$

under the action of I_3. Let us use the same notation for the lift of this action to the bundle of spinors. Now this action commutes with the Dirac operator, so we can ask

about the trace of I_3 on the space of Dirac zero modes. In fact, we will be able to glean this information in a simpler way which also reveals the electrical charges of the states we have found.

Note first that $\widetilde{I_3}$ defines a map $I_3^{(I)}$ which acts only on the I factor of $\widetilde{\mathrm{Ind}_2}$ in (3.12). Note that $I_3^{(I)}$ squares to -1, the map which is -1 on each fiber but trivial on the base. In fact, prior to this section we have only dealt with I. Global sections of $\widetilde{\mathrm{Ind}_2}$ have the form $\tilde{s}(t,m) = (t, s(t,m)) \in \mathbf{R} \times I$. The S^1 piece of the Dirac equation dictates the $t-$dependence of s to be $s(t,m) = e^{iQt/2}s(m)$, where s is a zero mode on I and Q is the electric charge. Sections of this form are clearly eigenstates of the electric charge operator $-2i\frac{\partial}{\partial t}$. Note that multiplication by a complex phase along the fibers is well-defined on I since it commutes with transition functions and is therefore independent of trivialization (in a given trivialization, s takes the form of a complex-valued function).

Now $\tilde{s}$ will descend to a global section on Ind_2 if

$$\begin{aligned}
\tilde{s}(t,m) &= \widetilde{I_3} \circ \tilde{s} \circ I_3^{-1}(t,m) \\
&= \widetilde{I_3} \circ \tilde{s}(t-\pi, I_3^{-1}m) \\
&= \widetilde{I_3}(t-\pi, e^{iQ(t-\pi)/2} I_3^{(I)} \circ s \circ I_3^{-1}m) \\
&= (t, e^{-iQ\pi/2} e^{iQt/2} I_3^{(I)} \circ s \circ I_3^{-1}m).
\end{aligned}$$

We saw $(I_3^{(I)})^2 = -1$ on I and since $I_3^{(I)}$ acts on the space of zero modes we can take s to have definite eigenvalue equal to r ($r = \pm i$). Putting $I_3^{(I)} \circ s \circ I_3^{-1}m = rs(m)$ and reinstating the $t-$dependence yields

$$r(-i)^q = 1. \tag{3.13}$$

In fact, this analysis holds for any odd power of I. For even powers, the bundle of zero modes is untwisted over the S^1 piece and therefore $I_3^{(I)}$ squares to the identity. Thus the eigenvalues r take the form $r = \pm 1$. The same condition (3.13) applies. Thus:

$$\begin{array}{llcc}
n \text{ odd} \Rightarrow & r = +i & \text{states have charge} & 1 \bmod 4 \\
 & r = -i & \text{''} & 3 \bmod 4 \\
n \text{ even} \Rightarrow & r = +1 & \text{states have charge} & 0 \bmod 4 \\
 & r = -1 & \text{''} & 2 \bmod 4.
\end{array}$$

In other words, every zero mode of I yields a physical monopole solution with electric charge dependent upon its $I_3^{(I)}$ eigenvalue.

How do we count the number of solutions with a given charge? We take the trace of $I_3^{(I)}$ on the space of zero sections. This is done for $n = 4$ (the fourth power of I) below. For $n = 3$ we remark that there is only one eigenvector, with eigenvalue $\pm i$, which must be the complex conjugate of the $n = -3$ eigenvector. These states are charge conjugates (note that 1 and -1 have different odd values mod 4). This argument tells us nothing for the even states, but for $n < 2$ there are no solutions anyway.

Let $L(I_3)$ denote the trace of $I_3^{(I)}$ restricted to the kernel of D on I^4. In order to compute $L(I_3)$ we need a noncompact variant of the Atiyah-Segal-Singer equivariant index theorem. We may argue exactly as we did in section 3.2 to obtain the following proposition. (See [15].)

Proposition 3.6.1

$$L(I_3) = lim_{L\to\infty}(lim_{s\to 0}\int_{r<L} \mathrm{tr}\tau I_3^{(I)} e^{-sD^2}dx + \int_0^\infty \int_{r=L} tre_0 \times \tau I_3^{(I)} De^{-sD^2} d\sigma ds).$$

The small s limit can be evaluated exactly as in the compact case. It localizes to a computation in a neighborhood of the fixed point set of I_3, but I_3 has no fixed points. Hence (see, for example, Section 6.3 of [16]),

$$lim_{L\to\infty}(lim_{s\to 0}\int_{r<L} \mathrm{tr}\tau I_3^{(I)} e^{-sD^2}dx) = 0.$$

We are left to compute

$$lim_{L\to\infty}\int_0^\infty \int_{r=L} tre_0\tau I_3^{(I)} De^{-sD^2} d\sigma ds.$$

We compute this defect term using the same semilocal parametrix we used in section 3.5. The action of $I_3^{(I)}$ changes the computation in two ways. First, it acts by $\pm e_1 e_2$ on the spinors. The sign is determined by the choice of the lift of the action of I_3 to the principal spin bundle. Secondly, $I_3^{(I)}$ introduces introduces a rotation of π in the fiber of the fibration (3.9). This enters the computation by having us evaluate the parametrix not on the diagonal, but along the diagonal on the base of the fibration and on $(\psi, \psi+\pi)$ in the fiber. For the k^{th} Fourier component, this is $(-1)^k$ times what one obtains by evaluating along the diagonal. So $I_3^{(I)}$ changes our computation by introducing fators of $e_1 e_2(-1)^k$. Once again trace identities make all terms vanish except

$$-\int_{RP^2}\sum_k\int_0^\infty tre_1e_2e_3e_1e_2(-1)^k e_3(e^L/2-1)(i[k+1]-(1/2-U^2/4)e_1e_2)\times$$
$$e^{\frac{s}{2u}(i[k+1]-(1/2-U^2/4)e_1e_2)^2}(4\pi s)^{-3/2}ds$$

$$=-(2\pi)\sum_k\int_0^\infty tr(-1)^k(e^L/2-1)(ik-(1/2-U^2/4)e_1e_2)e^{\frac{s}{2u}(ik-(1/2-U^2/4)e_1e_2)^2}\times$$
$$(4\pi s)^{-3/2}ds.$$

This expression is skew under the involution that interchanges the eigenspaces of e_1e_2 and sends $k\to -k$. Hence the trace is zero.

We can therefore conclude that for $n=4$

$$L(I_3)=0, \tag{3.14}$$

and for $n=-4$ we also find $L(I_3)=0$. So there are an equal number of charge 0 mod 4 and 2 mod 4 two-monopole states.

4. The BPS Spectrum

Let us briefly review the predictions of Seiberg and Witten and compare our results with their predictions. Of particular interest to us are the global symmetry properties of the theory. It will be an important check that the BPS states we find appear in predicted representations of the global symmetry group. Let us first exhibit the flavor symmetry discussed in [3] explicitly. For the purpose of determining the global symmetry, the Lagrangian contains a kinetic term for the hypermultiplets given by the bilinear form

$$M^\dagger M+\widetilde{M}^\dagger\widetilde{M},$$

where N_f flavor indices as well as the gauge group indices are suppressed. The chiral superfields M and $\widetilde{M}$ are in conjugate representations of the gauge group. However, since the gauge group is $SU(2)$, the fundamental and anti-fundamental are isomorphic, so a symmetry mixing M and $\widetilde{M}$ is permitted. The other term to be preserved in the Lagrangian is the coupling to the Higgs field $\widetilde{M}^T\Phi M$. If we define the $2N_f$-dimensional complex vector

$$V\equiv\begin{pmatrix} M+\widetilde{M}\\ i(M-\widetilde{M})\end{pmatrix},$$

then a symmetry $V \to AV$ must preserve $V^\dagger V$ and $V^T V$ i.e. $A \in U(2N_f)$ and $A \in O(2N_f; \mathbf{C})$. So $A^\dagger = A^{-1} = A^T$, and thus $A = A^*$; therefore A is in $O(2N_f)$ with real coefficients. At the quantum level, the parity in $O(2N_f)$ either reverses the sign of the electric charges (N_f odd), or is broken (N_f even). The relevant global symmetry group for states of a given charge is therefore $SO(2N_f)$.

As we discussed in section 2, the low-energy dynamics of monopoles and dyons is described by a supersymmetric sigma model. The Hilbert space decomposes into representations of $SO(2N_f)$. For magnetic charge $k = 1$, the bundle of zero modes is one-dimensional, and so the zero mode anti-commutation relations

$$\left\{\psi^i, \psi^j\right\} = \delta^{ij} \quad i = 1, \ldots, 2N_f$$

lead to spinorial representations of $SO(2N_f)$ as noted in [3]. For magnetic charge $k = 2$, the bundle is two-dimensional, and so the algebra that we must represent in terms of flavor properties is no longer a Clifford algebra but

$$\left\{\psi^i, \psi^{j\dagger}\right\} = \delta^{ij}, \tag{4.1}$$

where the ψ^i are now complex. This is just the usual annihilation and creation operator algebra. BPS states with magnetic charge two therefore fall into representations of $SO(2N_f)$ rather than its universal cover.

For $N_f = 0, 1, 2$, we found no BPS bound states with magnetic charge two, and the singularity structure proposed by Seiberg and Witten required no bound states. In the case $N_f = 0$, the semi-classical BPS spectrum is completely determined since there are no bound states for any magnetic charge $k > 1$. Such states would correspond to normalizable anti-holomorphic forms on a non-compact Calabi-Yau manifold, and there are no such forms.

For $N_f = 3$, we found a single bound state for each charge $(2, n)$ where n is any odd integer. These states are singlets under the $SO(6)$ flavor symmetry since they correspond to the Fock vacuum for the algebra (4.1) and its complex conjugate. In this case, the singularity structure of the moduli space can be analyzed by looking at the limit of three equal very massive quarks. Since the mass is large, there is a singularity in the semi-classical region of the moduli space where the scalar vacuum expectation value is large and the quarks (in the $\mathbf{3}$ of the $SU(3)$ flavor symmetry) become massless. In the strong coupling region one can integrate out the massive quarks semiclassically and relate the

singularities to the two known singularities – a monopole with $(n_m, n_e) = (1,0)$ and a dyon $(1,1)$ both in the $\mathbf{1}$ of $SU(3)$ – of the $N_f = 0$ quantum moduli space. In this way, the nonabelian global charges of the massless states at the singularities are calculated. These charges cannot change. One can then let the masses of the quarks go to zero and determine the representation theory under the full Spin(6) $= SU(4)$ symmetry. This requires the $\mathbf{3}$ and $\mathbf{1}$ to combine into a $\mathbf{4}$ singularity, while the other singularity (now a $\mathbf{1}$ under $SU(4)$) remains separate. For $SU(4)$ the center $\mathbf{Z}_4$ is determined by general consistency conditions to act as $e^{i\pi(n_m+2n_e)/2}$, yielding the condition for the $\mathbf{4}$ that $n_m + 2n_e = 1$, with minimal solution $(n_m, n_e) = (1,0)$. This just the standard $(1,0)$ BPS monopole. The singlet state is trivial under the center and thus has minimal solution $(n_m, n_e) = (2,1)$. As explained in [3], this state should be continuously connected to a BPS state which exists semi-classically, and we have found such a state.

The case of most interest is $N_f = 4$. This theory is conjectured by [3] to be self-dual under $SL(2,\mathbf{Z})$. However, the situation is somewhat different from the N=4 Yang-Mills theory which is also conjectured to be self-dual. For N=2, $N_f = 4$, the $SL(2,\mathbf{Z})$ is believed to act on the representation spaces of the Spin(8) global symmetry as well. The elementary hypermultiplets transform in the vector of Spin(8) (i.e. the fundamental of $SO(8)$), while the $(1,0)$ and $(1,1)$ monopole states transform in opposite chirality spinor representations. Although these representations are not isomorphic, they are all eight dimensional and permuted by the $\mathbf{S}_3$ which acts as outer automorphisms of Spin(8) (an inner automorphism would give an isomorphic representation). The trivial representation is trivial under this $\mathbf{S}_3$ as well. We will not further motivate this prediction, but will rather simply state the action of symmetry. The action of $SL(2,\mathbf{Z})$ on a state is to transform the monopole numbers in the usual way – $(n_m, n_e) \to (an_m + bn_e, cn_m + dn_e)$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$ – and the representation is transformed under $\rho \in \mathbf{S}_3$ as above, where ρ is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z})$ with entries mod 2 (the group of unit determinant matrices mod 2 is easily seen to be isomorphic to $\mathbf{S}_3$) acting by left multiplication on the representations $(0,0)$ = trivial o, $(0,1)$ = vector v, $(1,0)$ = spinor s, $(1,1)$ = spinor c, where all numbers are defined mod 2. Simply stated: the representations are determined by (n_m, n_e) mod 2.

The BPS spectrum contains a stable elementary electron with charge $(0,1)$. The self-duality conjecture then implies the existence of bound states for all charges (p,q) where p and q are relatively prime integers. Specifically, bound states with charge $(2,q)$ with q

odd must exist and appear in the vector representation of Spin(8). We indeed found such states from the bound state solution corresponding to the excitation of a single zero mode $\gamma^{i\dagger}|-4>$ on the vacuum with $U(1)$ charge -4, and its complex conjugate. The allowed electric charges are $4q+1$ for one state, and $4q+3$ for the other.

The BPS spectrum also contains neutrally stable heavy gauge bosons of charge $(0,2)$. If such states exist as discrete states in the theory then we should expect to see their partners under $SL(2,\mathbf{Z})$ with charges $(2p,2q)$ as discussed in [3]. These states must all be singlets under Spin(8). Further, the heavy gauge bosons are part of BPS multiplets with spins ≤ 1 unlike the electrons which are part of BPS multiplets with spins $\leq \frac{1}{2}$. From our computations, we have shown that four bosonic bound state solutions exist that are singlets under Spin(8). We found in (3.14) that the electric charges are $4q$ for two of the solutions and $4q+2$ for the remaining two. The existence of such states certainly implies that if the theory is self-dual then the heavy gauge bosons must exist as bound states at the threshold of decay into electrons. Two solutions are also required if one is to construct a BPS multiplet containing a vector particle. Our findings are certainly in accord with the proposed duality.

To further support the supposition that the two solutions at $n=\pm 4$ with a given electric charge are members of the same BPS multiplet, we can examine the difference in fermion number between the different bound state solutions. We begin by noting that in the models under consideration, the fermion number is always integral. The fermion number of a bound state solution comes from two sources [17]: the first contribution is from fermions in the effective action for pure N=2 Yang-Mills. Since the moduli space is a product, the action can always be written as the sum of two terms: $S_{eff}(k=1)$ and an interacting piece S_{int}. Quantizing the fermions from the first term gives us the usual four-dimensional BPS multiplet when acting on a spin zero vacuum. Our interest resides with the remaining fermions from S_{int}, and the difference in the fermion number between the $n=3$ and $n=4$ bound state solutions. Viewing these fermions as spinors, and noting that the index has the same sign for $n=3$ and $n=4$, we can conclude that the bound state solutions at $n=3$ and $n=4$ have the same chirality. Therefore, the difference in fermion number from this source is zero mod 2. The other contribution, from the matter fermions, clearly produces a difference in fermion number. Therefore, we can conclude that the bound states at $n=4$ differ in fermion number from the bound state at $n=3$ by one mod 2. Spin-statistics implies that the bound state at $n=3$ is bosonic, and so the two states at $n=4$ are fermionic. A BPS multiplet built on such a vacuum includes a vector particle as expected.

5. Conclusions

We have studied the question of whether bound states of monopoles and dyons with magnetic charge two exist in supersymmetric Yang-Mills coupled to matter. This problem was solved by computing the number of L^2 solutions of the Dirac equation for bundle-valued spinors over the two-monopole moduli space. For $N_f < 3$, no bound states exist, while for $N_f = 3$, there is a single bound state for every odd value of the electric charge, which is a singlet under the $SU(4)$ symmetry group. For $N_f = 4$, there is a bound state for each odd value of the electric charge. These bound states are in the vector representation of the Spin(8) flavor symmetry. There are also two bound states for each even value of the electric charge, which are singlets under the flavor group. Our findings provide dynamical evidence for the moduli space structure proposed by Seiberg and Witten – specifically, for the $N_f = 4$ conjectured self-duality. To show the BPS spectrum of the $N_f = 4$ theory is truly $SL(2, \mathbf{Z})$ invariant, similiar calculations are needed for higher magnetic charge. The main obstacle is the limited information about the metric for the higher charge monopole moduli spaces. Recently, the asymptotic metric on the k-monopole moduli space for the region where all k monopoles are far apart has been described [18]. However, an understanding of the metric at the boundaries of codimension one would be desirable to extend computations of this type.

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